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Deformations of Classical Geometries and Integrable Systems

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Abstract

A generalization of the notion of a (pseudo-) Riemannian space is proposed in a framework of noncommutative geometry. In particular, there are parametrized families of generalized Riemannian spaces which are deformations of classical geometries. We also introduce harmonic maps on generalized Riemannian spaces into Hopf algebras and make contact with integrable models in two dimensions.

1 Introduction

As a classical geometry we understand an n-dimensional Riemannian[1] space M which consists of a smooth orientable manifold M and a metric tensor field

$$g = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes_{\mathcal{A}} \mathrm{d}x^\nu \tag{1}$$

where $\mathcal{A}$ is the algebra of smooth function on M. The metric induces a Hodge operator

$$\star \;:\; \Lambda^r(M) \to \Lambda^{n-r}(M) \tag{2}$$

where $\Lambda^r(M)$ is the space of differential r-forms on M. From the action of the Hodge operator we recover the (inverse) metric components with respect to the coordinates x^μ as follows,

$$g^{\mu\nu} = \star^{-1}(\mathrm{d}x^\mu \wedge \star\, \mathrm{d}x^\nu)\;. \tag{3}$$

[1]Here and in the following 'Riemannian' includes pseudo-Riemannian, i.e., the case of an indefinite metric.

A generalization of classical geometries is obtained by generalizing the concept of differential forms, accompanied with a suitable generalization of the Hodge operator. The algebra of (ordinary) differential forms is then replaced by some 'noncommutative' differential algebra on M. Essentially, this means that we keep all the basic formulas of the classical differential calculus but dispense with commutativity of functions and differentials. A further generalization of geometries consists in replacing the underlying space M, or rather the (suitably restricted) algebra of functions on it, by some noncommutative associative algebra $\mathcal{A}$. All this will be made more precise in section 2.

Given a generalized Riemannian space, one can consider analogues of physical models and dynamical systems on it. Of particular interest are generalized geometries and models which are deformations of classical geometries and models in the sense that they depend on some parameter in such a way that the basic algebraic relations become the classical ones when the parameter tends to a certain value. We then have the chance to study models which are 'close' to known models of physical relevance. Section 3 is devoted to corresponding generalizations of harmonic maps into groups (or Hopf algebras), which are also known as (a class of) σ-models or principal chiral models. This is based on our previous work [1-5]. In subsection 3.3 we make an attempt to generalize the latter to *non*commutative algebras. We have to stress, however, that this is more a report on work in progress than something which has reached a satisfactory status. Section 4 contains some conclusions.

2 Generalizations of classical geometries

Let $\mathcal{A}$ be an associative algebra with unit element $\mathbb{1}$. A *differential calculus* on $\mathcal{A}$ consists of a differential algebra $\Omega(\mathcal{A})$ and an operator d which shares some basic properties with the exterior derivative of the ordinary differential calculus on manifolds. A *differential algebra* is a $\mathbb{Z}$-graded associative algebra (over $\mathbb{R}$, respectively $\mathbb{C}$)

$$\Omega(\mathcal{A}) = \bigoplus_{r\geq 0} \Omega^r(\mathcal{A}) \tag{4}$$

where the spaces $\Omega^r(\mathcal{A})$ are $\mathcal{A}$-bimodules and $\Omega^0(\mathcal{A}) = \mathcal{A}$. The operator d is a linear[2] map

$$\mathrm{d} \; : \quad \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A}) \tag{5}$$

with the properties

$$\mathrm{d}^2 = 0 \tag{6}$$

$$\mathrm{d}(w\, w') = (\mathrm{d}w)\, w' + (-1)^r\, w\, \mathrm{d}w' \tag{7}$$

where $w \in \Omega^r(\mathcal{A})$ and $w' \in \Omega(\mathcal{A})$. The last relation is known as the (generalized) *Leibniz rule*. We also require $\mathbb{1}\, w = w\, \mathbb{1} = w$ for all elements $w \in \Omega(\mathcal{A})$. The identity $\mathbb{1}\mathbb{1} = \mathbb{1}$

[2]Here and in the following *linear* means linear over the respective field which is $\mathbb{R}$ or $\mathbb{C}$ in the cases under consideration.

then implies $\mathrm{d}\mathbb{1} = 0$. Furthermore, it is assumed that d generates the spaces $\Omega^r(\mathcal{A})$ for $r > 0$ in the sense that $\Omega^r(\mathcal{A}) = \mathcal{A}\,\mathrm{d}\Omega^{r-1}(\mathcal{A})\,\mathcal{A}$.

2.1 Commutative algebras with noncommutative differential calculi and the Hodge operator

Let $\mathcal{A}$ be a *commutative* algebra, freely generated by elements x^μ, $\mu = 1, \ldots, n$. A differential calculus $(\Omega(\mathcal{A}), \mathrm{d})$ is called *n-dimensional* if
(1) $\mathrm{d}x^\mu$ is a left and also a right $\mathcal{A}$-module basis of $\Omega^1(\mathcal{A})$,
(2) $\Omega^r(\mathcal{A}) = \{0\}$ for $r > n$, but $\Omega^n(\mathcal{A}) \neq \{0\}$,
(3) $\dim \Omega^r(\mathcal{A}) = \dim \Omega^{n-r}(\mathcal{A})$ as left as well as right $\mathcal{A}$-modules ($r = 0, \ldots, n$).

In the following we consider a (freely generated) commutative algebra $\mathcal{A}$ with an n-dimensional differential calculus $(\Omega(\mathcal{A}), \mathrm{d})$. A *generalized Hodge operator* is a linear invertible map

$$\star\, : \; \Omega^r \to \Omega^{n-r} \qquad r = 0, \ldots, n \tag{8}$$

such that[3]

$$\star\, (w\, f) = f \,\star\, w \qquad \forall f \in \mathcal{A},\, w \in \Omega^r(\mathcal{A}) \; . \tag{9}$$

Defining $\star$ on a basis of r-forms, this covariance property allows us to calculate its action on any r-form. According to (3) we should be most interested in the action of $\star$ on 1-forms and n-forms. We call $(\mathcal{A}, \Omega(\mathcal{A}), \mathrm{d}, \star)$ an n-dimensional *generalized Riemannian space.*

Example. Let $\mathcal{A}$ be the algebra of functions on the lattice $\mathbb{Z}^n$ with the n-dimensional differential calculus determined by

$$[\mathrm{d}x^\mu, x^\nu] = \delta^{\mu\nu}\, \mathrm{d}x^\nu \tag{10}$$

in terms of the canonical coordinates x^μ on $\mathbb{Z}^n$ (cf [6]). As a consequence, we have

$$\mathrm{d}f = \sum_{\mu=1}^{n} [f(x^1, \ldots, x^{\mu-1}, x^\mu + 1, x^{\mu+1}, \ldots, x^n) - f(x^1, \ldots, x^n)]\, \mathrm{d}x^\mu \tag{11}$$

and

$$\mathrm{d}x^\mu\, \mathrm{d}x^\nu = -\mathrm{d}x^\nu\, \mathrm{d}x^\mu \; . \tag{12}$$

This familiar anticommutativity of differentials does not extend to general 1-forms, however, as in the ordinary calculus of differential forms. Let $\epsilon_{\mu_1 \ldots \mu_n}$ be totally antisymmetric with $\epsilon_{1 \ldots n} = 1$ and $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \ldots, -1)$. We define

$$\star\, (\mathrm{d}x^{\mu_1} \cdots \mathrm{d}x^{\mu_r}) := \frac{1}{(n-r)!}\, \eta^{\mu_1\nu_1} \cdots \eta^{\mu_r\nu_r}\, \epsilon_{\nu_1 \ldots \nu_r \kappa_1 \ldots \kappa_{n-r}}\, \mathrm{d}x^{\kappa_1} \cdots \mathrm{d}x^{\kappa_{n-r}} \; . \tag{13}$$

What we have here is a discrete version of the n-dimensional Minkowski space. Note that

$$\mathrm{d}x^\mu \star \mathrm{d}x^\nu = \star\, \eta^{\mu\nu} \; . \tag{14}$$

[3]The 'twist' in (9) is dictated by certain examples (which do not work with the alternative rule $\star(f\,w) = f\,\star\,w$). Note that the inverse of $\star$ satisfies $\star^{-1}(f\,w) = (\star^{-1}w)\,f$.

In terms of the rescaled coordinates $x^{\mu'} := \ell^\mu x^\mu$ with constants $\ell^\mu > 0$, (10) becomes

$$[\mathrm{d}x^{\mu'}, x^{\nu'}] = \ell^\mu \, \delta^{\mu\nu} \, \mathrm{d}x^{\nu'} \ . \tag{15}$$

Ignoring the origin of the primed coordinates, this is a deformation of the algebraic relations of the ordinary differential calculus on $\mathbb{R}^n$ (where differentials and functions commute). For each coordinate a contraction $\ell^\mu \to 0$ can then be performed. In the new coordinates, the metric components are

$$g^{\mu'\nu'} := \star^{-1}(\mathrm{d}x^{\mu'} \star \mathrm{d}x^{\nu'}) = \ell^\mu \, \ell^\nu \, \eta^{\mu\nu} \tag{16}$$

and are thus witness to the rescaling. A less trivial coordinate transformation is given by

$$y^\mu := (q^\mu)^{x^{\mu'}/\ell^\mu} \tag{17}$$

with $q^\mu \in \mathbb{C} \setminus \{0, 1\}$ and not a root of unity. This implies

$$\mathrm{d}y^\mu = \frac{q^\mu - 1}{\ell^\mu} \, y^\mu \, \mathrm{d}x^{\mu'} \ , \qquad \mathrm{d}x^{\mu'} \, y^\mu = q^\mu \, y^\mu \, \mathrm{d}x^{\mu'} \tag{18}$$

and turns (15) into the 'quantum plane' relations

$$\mathrm{d}y^\mu \, y^\mu = q^\mu \, y^\mu \, \mathrm{d}y^\mu \tag{19}$$

(see also [7]). For different indices $\mu \neq \nu$, $\mathrm{d}y^\mu$ and y^ν simply commute. The components of the metric in the new coordinates y^μ are

$$g^{\mu\nu} := \star^{-1}(\mathrm{d}y^\mu \star \mathrm{d}y^\nu) = (q^\mu - 1) \, (q^\nu - 1) \, y^\mu \, y^\nu \, \eta^{\mu\nu} \ . \tag{20}$$

One might expect that the inverse $g_{\mu\nu}$ defines an invariant object via $g_{\mu\nu} \, \mathrm{d}y^\mu \otimes_{\mathcal{A}} \mathrm{d}y^\nu$, $\mathrm{d}y^\mu \otimes_{\mathcal{A}} g_{\mu\nu} \, \mathrm{d}y^\nu$ or $\mathrm{d}y^\mu \otimes_{\mathcal{A}} \mathrm{d}y^\nu \, g_{\mu\nu}$. However, none of these expressions is equal to $\eta_{\mu\nu} \, \mathrm{d}x^\mu \otimes_{\mathcal{A}} \mathrm{d}x^\nu$, but differs by a factor which is a power of q. $\diamondsuit$

Using the Hodge operator, we define a scalar product on Ω^1 by setting

$$(\alpha, \beta) := \star^{-1}(\alpha \star \beta) \ . \tag{21}$$

From (9) and the corresponding formula for the inverse of $\star$ we obtain

$$(\alpha, \beta \, f) = (\alpha \, f, \beta) \, , \qquad (f \, \alpha, \beta) = (\alpha, \beta) \, f = f \, (\alpha, \beta) \tag{22}$$

(since $\mathcal{A}$ is assumed to be commutative). The components of the scalar product are

$$g^{\mu\nu} := (\mathrm{d}x^\mu, \mathrm{d}x^\nu) \ . \tag{23}$$

Let $y^\mu \in \mathcal{A}$. Then

$$dy^\mu = (\hat{\partial}_\nu y^\mu) \, \mathrm{d}x^\nu \tag{24}$$

with generalized partial derivatives $\hat{\partial}_\nu$. We call y^μ 'coordinates' if $\hat{\partial}_\nu y^\mu$ is invertible. From the above properties of the scalar product we get

$$g^{\mu'\nu'} = (\mathrm{d}y^\mu, \mathrm{d}y^\nu) = \hat{\partial}_\kappa y^\mu \, (\mathrm{d}x^\kappa, \hat{\partial}_\lambda y^\nu \, \mathrm{d}x^\lambda) \; . \tag{25}$$

Let us now assume that the scalar product is *symmetric*, i.e., $(\alpha, \beta) = (\beta, \alpha)$ for all 1-forms α, β, which means

$$\alpha \star \beta = \beta \star \alpha \; . \tag{26}$$

In this case we have also

$$(\alpha, f\,\beta) = f\,(\alpha, \beta) \; . \tag{27}$$

and thus

$$g^{\mu'\nu'} = \hat{\partial}_\kappa y^\mu \, \hat{\partial}_\lambda y^\nu \, (\mathrm{d}x^\kappa, \mathrm{d}x^\lambda) = \hat{\partial}_\kappa y^\mu \, \hat{\partial}_\lambda y^\nu \, g^{\kappa\lambda} \; . \tag{28}$$

To construct a tensor field from these components, the usual tensor product $\otimes_{\mathcal{A}}$ is not the right one as long as functions do not commute with differentials (see the example above). In the case of a commutative algebra there is also a tensor product, denoted as $\otimes_L$, which (besides bilinearity over $\mathbb{R}$, respectively $\mathbb{C}$) satisfies

$$(f\,\alpha) \otimes_L (h\,\beta) = f\,h\,(\alpha \otimes_L \beta) \; . \tag{29}$$

Then

$$g := g_{\mu\nu} \, dx^\mu \otimes_L dx^\nu \tag{30}$$

is a tensorial object.

2.2 Noncommutative algebras and the Hodge operator

The covariance property for the Hodge operator, as formulated in (9), is not compatible with a *non*commutative algebra $\mathcal{A}$. A modification is needed. Let † be an involution of $\mathcal{A}$. We generalize the covariance rule as follows,[4]

$$\star\,(w\,f) = f^\dagger \star w \tag{31}$$

so that

$$\star\,(w\,(fh)) = (fh)^\dagger \star w = h^\dagger\, f^\dagger \star w = h^\dagger \star (wf) = \star\,((wf)\,h) \; . \tag{32}$$

Again, we assume that the Hodge operator is an invertible map $\Omega^r(\mathcal{A}) \to \Omega^{n-r}(\mathcal{A})$ for some $n \in \mathbb{N}$. For its inverse (31) implies

$$\star^{-1}\,(f\,w) = (\star^{-1} w)\, f^\dagger \; . \tag{33}$$

As a consequence, the scalar product on $\Omega^1(\mathcal{A})$, defined again by (21), satisfies

$$(\alpha, \beta\, f) = (\alpha\, f^\dagger, \beta) \, , \qquad (f\,\alpha, \beta) = (\alpha, \beta)\, f^\dagger \; . \tag{34}$$

[4]Note that this rule does not reduce to our previous rule in the case of a commutative algebra $\mathcal{A}$ when the involution acts nontrivially on $\mathcal{A}$.

Let us now assume that † extends to an involution of $\Omega(\mathcal{A})$ so that

$$(w\, w')^\dagger = w'^\dagger\, w^\dagger\ . \tag{35}$$

We still have to define how the exterior derivative d interacts with the involution. Here we adopt the following rule

$$(\mathrm{d}w)^\dagger = (-1)^{r+1}\, \mathrm{d}(w^\dagger) \tag{36}$$

for $w \in \Omega^r(\mathcal{A})$ (cf [8], for example).

We can now consistently impose the condition

$$(\star w)^\dagger = \star^{-1}(w^\dagger) \tag{37}$$

since

$$(\star\,(w\,f))^\dagger = (f^\dagger\,\star w)^\dagger = (\star\, w)^\dagger\, f = [\star^{-1}(w^\dagger)]\, f = \star^{-1}(f^\dagger w^\dagger) = \star^{-1}[(wf)^\dagger]\ . \tag{38}$$

3 Generalized harmonic maps into matrix Hopf algebras

Let H be a matrix Hopf algebra (cf [9], in particular). This is a Hopf algebra generated by elements $\mathbf{a}^i{}_j$, $i,j = 1,\ldots,N$. The coproduct $\Phi : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ is given by

$$\Phi(\mathbf{a}^i{}_j) = \mathbf{a}^i{}_k \otimes \mathbf{a}^k{}_j \tag{39}$$

using the summation convention. The antipode S satisfies

$$S(\mathbf{a}^i{}_k)\, \mathbf{a}^k{}_j = \delta^i_j\, \mathbb{1} = \mathbf{a}^i{}_k\, S(\mathbf{a}^k{}_j)\ . \tag{40}$$

In terms of the $N\times N$ matrix $\mathbf{a} = (\mathbf{a}^i{}_j)$ we have $\Phi(\mathbf{a}) = \mathbf{a}\otimes\mathbf{a}$ and

$$S(\mathbf{a})\, \mathbf{a} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} = \mathbf{a}\, S(\mathbf{a}) \tag{41}$$

in matrix notation. Let $(\mathcal{A}, \Omega(\mathcal{A}), \mathrm{d}, \star)$ be a generalized geometry in the sense of the preceeding section and let us assume that the entries of $\mathbf{a}$ are constructed from elements of $\mathcal{A}$. The matrix of 1-forms

$$A := S(\mathbf{a})\, \mathrm{d}\mathbf{a} \tag{42}$$

then satisfies the identity

$$F := \mathrm{d}A + AA = 0\ . \tag{43}$$

The field equation

$$\mathrm{d} \star A = 0 \tag{44}$$

now defines a *generalized harmonic map* into a matrix Hopf algebra.[5] Note that we do not need the full Hodge operator here, but only its restriction to 1-forms, i.e., $\star : \Omega^1(\mathcal{A}) \to \Omega^{n-1}(\mathcal{A})$.

A *generalized conserved current* of a generalized harmonic map is a 1-form J which satisfies

$$\mathrm{d} \star J = 0 \tag{45}$$

as a consequence of the field equation (44). We call a generalized harmonic map (completely) *integrable* if there is an infinite set of independent[6] conserved currents.

3.1 Integrable 2-dimensional generalized harmonic maps on commutative algebras

For 2-dimensional classical σ-models there is a construction of an infinite tower of conserved currents [12]. This has been generalized in [1-4] to harmonic maps on ordinary (topological) spaces, but with *non*commutative differential calculi, and values in a matrix group. In the following, we briefly recall the essential steps of our construction.

Let us consider a generalized harmonic map on a 2-dimensional generalized Riemannian space (in the sense of subsection 2.1) which satisfies the symmetry condition (26) and furthermore, for $\alpha \in \Omega^1(\mathcal{A})$,

$$\mathrm{d}\alpha = 0 \quad \Rightarrow \quad \alpha = \star\star\, \mathrm{d}\chi \tag{46}$$

with a function χ.

Let us start with the $N \times N$ matrix

$$\chi^{(0)} := \begin{pmatrix} \mathbb{1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbb{1} \end{pmatrix} . \tag{47}$$

Then

$$J^{(1)} := D\chi^{(0)} = (\mathrm{d} + A)\, \chi^{(0)} = A \tag{48}$$

is conserved as a consequence of the field equation. Using (46) this implies

$$J^{(1)} = \star\, \mathrm{d}\chi^{(1)} \tag{49}$$

[5]We may also call this a *generalized principal chiral model* or a *generalized σ-model* (see [10], for example). Some related 'noncommutative examples' can be found in [11].

[6]A convenient notion of independence in this context still has to be found. For example, in the case of classical models where integration is defined and the conserved currents lead to conserved charges, it may happen that a charge is a polynomial in some other charges. In an extreme case, we could get an infinite tower of conserved charges as the set of polynomials in a single charge. We would not like to talk about complete integrability in such a case.

with an $N \times N$ matrix $\chi^{(1)}$. Now

$$J^{(2)} := D\chi^{(1)} \tag{50}$$

is also conserved,

$$\mathrm{d} \star J^{(2)} = \mathrm{d} \star D\chi^{(1)} = D \star \mathrm{d}\chi^{(1)} = DJ^{(1)} = D^2\chi^{(0)} = F = 0\ , \tag{51}$$

since $\mathrm{d} \star D = D \star \mathrm{d}$ on $N \times N$ matrices with entries in $\mathcal{A}$. The latter follows from (9), (44) and (26) (cf [1,2]). Again, (46) implies

$$J^{(2)} := \star\, \mathrm{d}\chi^{(2)} \tag{52}$$

with an $N \times N$ matrix $\chi^{(2)}$ of elements of $\mathcal{A}$. Now

$$J^{(3)} := D\chi^{(2)} \tag{53}$$

is another $N \times N$ matrix of conserved currents, and so forth. In this way we obtain an infinite set of (matrices of) conserved currents. There is no guarantee, however, that all these currents are really independent. It can happen, as in the case of the free linear wave equation on two-dimensional Minkowski space, that the higher conserved charges are just polynomials in a finite number of independent ones.

In this subsection we have considered a *commutative* algebra $\mathcal{A}$, but with a *non*commutative differential calculus. Even in this case a huge set of possibilities for integrable models arises and several examples have already been elaborated [1-4].

Example. We recall the following example from [1] (see also [2-5]). Let $\mathcal{A}$ be the algebra of functions $f(t,x)$ on $\mathbb{R} \times \mathbb{Z}$ which are smooth in the first argument. A differential calculus on $\mathcal{A}$ is then determined by the relations

$$[\mathrm{d}t, t] = 0\,, \quad [\mathrm{d}x, x] = \ell\, \mathrm{d}x\,, \quad [\mathrm{d}t, x] = [\mathrm{d}x, t] = 0\ . \tag{54}$$

A Hodge operator, restricted to 1-forms, is given by[7]

$$\star\, \mathrm{d}t = -\mathrm{d}x\,, \qquad \star\, \mathrm{d}x = -\mathrm{d}t\ . \tag{55}$$

The differential calculus and the Hodge operator satisfy the conditions (26) and (46). Let $\mathbf{a} = e^{-u}$ with a function $u(t,x)$, so we consider only the case where $N = 1$. Then, using (9) and (11), the field equation $\mathrm{d} \star A = 0$ turns out to be the equation of the nonlinear Toda lattice,

$$\ddot{u}_k + \frac{1}{\ell^2}(e^{u_k - u_{k+1}} - e^{u_{k-1} - u_k}) = 0 \tag{56}$$

where $u_k(t) := u(t, k\,\ell)$. In the limit as $\ell \to 0$ the generalized geometry tends to that of the 2-dimensional Minkowski space and the above field equation becomes the linear wave equation. We refer to the references mentioned above for details and also for matrix generalizations of the Toda lattice (i.e., $N > 1$). $\diamondsuit$

A new example is presented in the following subsection.

[7] Note that the parameter ℓ does not appear in these relations. Hence they are not obtained by a simple coordinate rescaling $x \mapsto x/\ell$ from the $\ell = 1$ formulas. This has to be distinguished from what we did in the example in subsection 2.1.

3.2 Another example

In [6] we found in particular the following differential calculus,

$$[\mathrm{d}t, t] = b\,\mathrm{d}t\,, \quad [\mathrm{d}t, x] = b\,\mathrm{d}x\,, \quad [\mathrm{d}x, t] = b\,\mathrm{d}x\,, \quad [\mathrm{d}x, x] = -\frac{a^2}{b}\,\mathrm{d}t \tag{57}$$

with real constants $a, b \neq 0$. In terms of the complex variable $z = t/b + i\,x/a$, the above commutation relations read

$$[\mathrm{d}z, z] = 2\,\mathrm{d}z\,, \quad [\mathrm{d}z, \bar{z}] = 0\,, \quad [\mathrm{d}\bar{z}, z] = 0\,, \quad [\mathrm{d}\bar{z}, \bar{z}] = 2\,\mathrm{d}\bar{z} \tag{58}$$

where $\bar{z}$ is the complex conjugate of z. In the complex coordinates $z, \bar{z}$ we thus have a two-dimensional lattice differential calculus (cf the example in subsection 2.1). The relations (57) extend to arbitrary functions $f_t(x) := f(t,x)$ as follows,

$$\begin{aligned}
\mathrm{d}t\, f_t &= C_x f_{t+b}\,\mathrm{d}t + \frac{b}{a}\,S_x f_{t+b}\,\mathrm{d}x \\
\mathrm{d}x\, f_t &= -\frac{a}{b}\,S_x f_{t+b}\,\mathrm{d}t + C_x f_{t+b}\,\mathrm{d}x \\
f_t\,\mathrm{d}t &= \mathrm{d}t\, C_x f_{t-b} - \frac{b}{a}\,\mathrm{d}x\, S_x f_{t-b} \\
f_t\,\mathrm{d}x &= \frac{a}{b}\,\mathrm{d}t\, S_x f_{t-b} + \mathrm{d}x\, C_x f_{t-b}
\end{aligned} \tag{59}$$

where the operators C_x, S_x are defined by

$$(C_x f)(x) := \frac{1}{2}\,[f(x+ia) + f(x-ia)] \tag{60}$$

$$(S_x f)(x) := \frac{1}{2i}\,[f(x+ia) - f(x-ia)] \tag{61}$$

acting on a function of x. They satisfy

$$C_x(fh) = (C_x f)(C_x h) - (S_x f)(S_x h) \tag{62}$$

$$S_x(fh) = (S_x f)(C_x h) + (C_x f)(S_x h) \tag{63}$$

and

$$C_x^2 f + S_x^2 f = f\;. \tag{64}$$

Furthermore, we have

$$\mathrm{d}f_t = \frac{1}{b}\,(C_x f_{t+b} - f_t)\,\mathrm{d}t + \frac{1}{a}\,(S_x f_{t+b})\,\mathrm{d}x\;. \tag{65}$$

Using $\mathrm{d}t\,\mathrm{d}t = \mathrm{d}x\,\mathrm{d}x = \mathrm{d}t\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}t = 0$, which follows from (57) by application of the exterior derivative d, a Hodge operator which satisfies (26) is given by

$$\star\;\mathrm{d}t = \sigma\,\mathrm{d}t + \frac{b}{a}\,\kappa\,\mathrm{d}x\,, \qquad \star\,\mathrm{d}x = \frac{a}{b}\,\kappa\,\mathrm{d}t - \sigma\,\mathrm{d}x \tag{66}$$

with constants κ, σ. When $\kappa^2 + \sigma^2 = 1$, it has the property $\star\star\, w = w$ for all 1-forms w. Together with the property of the differential calculus that every closed 1-form is exact[8], this implies that (46) holds. Furthermore, a direct calculation shows that also (26) is satisfied. The above constraint for the constants κ, σ is solved by writing $\sigma = \sin\theta$ and $\kappa = \cos\theta$ with a parameter θ.

With $\mathbf{a} = e^{-u_t}$ we obtain

$$\begin{aligned} A = e^{u_t}\,\mathrm{d}e^{-u_t} &= \frac{1}{b}\,(e^{u_t} C_x e^{-u_{t+b}} - 1)\,\mathrm{d}t + \frac{1}{a}\,(e^{u_t} S_x e^{-u_{t+b}})\,\mathrm{d}x \\ &= \mathrm{d}t\,\frac{1}{b}\,(e^{-u_t} C_x e^{u_{t-b}} - 1) - \mathrm{d}x\,\frac{1}{a}\,(e^{-u_t} S_x e^{u_{t-b}}) \,. \end{aligned} \tag{67}$$

Application of the Hodge operator leads to

$$\begin{aligned} \star\, A &= -\frac{1}{b}\,[\sin\theta + e^{-u_t}\,\sin(a\,\partial_x - \theta)\,e^{u_{t-b}}]\,\mathrm{d}t \\ &\quad + \frac{1}{a}\,[e^{-u_t}\,\cos(a\,\partial_x - \theta)\,e^{u_{t-b}} - \cos\theta]\,\mathrm{d}x \end{aligned} \tag{68}$$

and the field equation $\mathrm{d}\star A = 0$ takes the form

$$e^{u_t}\,\cos(a\,\partial_x - \theta)\,e^{-u_{t+b}} = e^{-u_t}\,\cos(a\,\partial_x - \theta)\,e^{u_{t-b}} \tag{69}$$

which, admittedly, is a rather unfamiliar equation.

3.3 Generalization to noncommutative algebras

In order to generalize the construction of conservation laws to *non*commutative algebras $\mathcal{A}$, we impose some conditions in addition to those already introduced in subsection 2.2. In particular, we make the assumption that for each $r = 0, \ldots, n$ there is a constant $\epsilon_r \neq 0$ such that[9]

$$\star\star w = \epsilon_r\, w \qquad \forall w \in \Omega^r \tag{70}$$

respectively,

$$\star\; w = \epsilon_r^\dagger\, \star^{-1} w \,. \tag{71}$$

Applying the involution and using (37) we find

$$\frac{1}{\epsilon_r^\dagger}\,\star(w^\dagger) = \star^{-1}(w^\dagger) = (\star w)^\dagger = \epsilon_r\,(\star^{-1} w)^\dagger = \epsilon_r\,\star(w^\dagger) \tag{72}$$

[8] The proof is simple. It is a slight variation of the proof of a Lemma in [4].

[9] An apparently weaker assumption would be: for each $w \in \Omega^r(\mathcal{A})$ exists a constant ϵ_w such that $\star\star\, w = \epsilon_w\, w$. However, this reduces to our previous assumption as follows. Since $\star\star$ is linear, we have $\epsilon_{w+w'}\,(w + w') = \star\star\,(w + w') = \star\star\, w + \star\star\, w' = \epsilon_w w + \epsilon_{w'} w'$ and thus $(\epsilon_{w+w'} - \epsilon_w) w = (\epsilon_{w+w'} - \epsilon_{w'}) w'$ for arbitrary $w, w' \in \Omega(\mathcal{A})$. If w, w' are linearly independent, then $\epsilon_w = \epsilon_{w'} = \epsilon_{w+w'}$. Furthermore, $c\,\epsilon_w w = c\,\star\star w = \star\star(c\,w) = \epsilon_{cw}\, c\, w$ implies $\epsilon_{cw} = \epsilon_w$ where $c \in \mathbb{C}$. It follows that ϵ is constant on $\Omega^r(\mathcal{A})$, i.e., $\epsilon_w = \epsilon_r\ \forall\, w \in \Omega^r(\mathcal{A})$.

and thus

$$\epsilon_r^\dagger = \frac{1}{\epsilon_r} \; . \tag{73}$$

Instead of the symmetry condition (26) we impose the condition

$$(\alpha \star \beta)^\dagger = \epsilon_n^\dagger \, \beta \star \alpha \tag{74}$$

where $\alpha, \beta \in \Omega^1(\mathcal{A})$. This is consistent with (31) since

$$[\alpha \star (\beta \, f)]^\dagger = [\alpha \, f^\dagger \, \star \beta]^\dagger = \epsilon_n^\dagger \, \beta \star (\alpha \, f^\dagger) = \epsilon_n^\dagger \, (\beta \, f) \star \alpha \; . \tag{75}$$

As a consequence, we find

$$\begin{aligned} (\alpha, \beta)^\dagger &= [\star^{-1}(\alpha \star \beta)]^\dagger = [\star^{-1} \, \epsilon_n \, (\beta \star \alpha)^\dagger]^\dagger = \epsilon_n \, \star (\beta \star \alpha) = \epsilon_n \, \epsilon_n^\dagger \, (\beta, \alpha) \\ &= (\beta, \alpha) \; . \end{aligned} \tag{76}$$

A crucial step in the construction of conserved currents for harmonic maps on commutative algebras in subsection 3.1 is the identity $\mathrm{d} \star D = D \star \mathrm{d}$. A suitable generalization is now obtained as follows. First, we have

$$(\mathrm{d} \star \mathrm{d}\chi^i{}_j)^\dagger = \mathrm{d}(\star \mathrm{d}\chi^i{}_j)^\dagger = \mathrm{d} \star^{-1} (\mathrm{d}\chi^i{}_j)^\dagger = -\mathrm{d} \star^{-1} \mathrm{d}(\chi^i{}_j)^\dagger = -\epsilon_1 \, \mathrm{d} \star \mathrm{d}(\chi^i{}_j)^\dagger \tag{77}$$

using (36), (37), again (36) and then (71). Furthermore,

$$[\mathrm{d}(\chi^k{}_j)^\dagger \star A^i{}_k]^\dagger = \epsilon_n^\dagger \, A^i{}_k \star \mathrm{d}(\chi^k{}_j)^\dagger \tag{78}$$

using (74). Hence

$$\begin{aligned} \mathrm{d} \star D\chi^i{}_j &= \mathrm{d} \star (\mathrm{d}\chi^i{}_j + A^i{}_k \, \chi^k{}_j) = \mathrm{d} \star \mathrm{d}\chi^i{}_j + \mathrm{d}((\chi^k{}_j)^\dagger \star A^i{}_k) \\ &= \mathrm{d} \star \mathrm{d}\chi^i{}_j + \mathrm{d}(\chi^k{}_j)^\dagger \star A^i{}_k + (\chi^k{}_j)^\dagger \mathrm{d} \star A^i{}_k \\ &= [(\mathrm{d} \star \mathrm{d}\chi^i{}_j)^\dagger + (\mathrm{d}(\chi^k{}_j)^\dagger \star A^i{}_k)^\dagger]^\dagger \\ &= [-\epsilon_1 \, \mathrm{d} \star \mathrm{d}(\chi^i{}_j)^\dagger + \epsilon_n^\dagger \, A^i{}_k \star (\chi^k{}_j)^\dagger]^\dagger \end{aligned} \tag{79}$$

using $\mathrm{d} \star A = 0$. Consequently, if $\epsilon_1 = -\epsilon_n^\dagger$ we have

$$\mathrm{d} \star D(\chi^\dagger) = -(\epsilon_1^\dagger \, D \star \mathrm{d}\chi)^\dagger \; . \tag{80}$$

For $n = 2$ and assuming (46), the construction of conservation laws now generalizes to the case of a *non*commutative algebra $\mathcal{A}$.

Example. Let $\mathcal{A}$ be the Heisenberg algebra with the two generators q and p satisfying

$$[q, p] = i \, \hbar \; . \tag{81}$$

In the simplest differential calculus on $\mathcal{A}$ we have

$$[\mathrm{d}q, f] = 0 \, , \qquad [\mathrm{d}p, f] = 0 \tag{82}$$

for all $f \in \mathcal{A}$ (see also [13]). It follows that

$$\mathrm{d}f = (\hat{\partial}_q f)\,\mathrm{d}q + (\hat{\partial}_p f)\,\mathrm{d}p \tag{83}$$

where the generalized partial derivatives are given by

$$\hat{\partial}_q f := -\frac{1}{i\hbar}[p,f]\,, \qquad \hat{\partial}_p f := \frac{1}{i\hbar}[q,f]\,. \tag{84}$$

Moreover, the relations (82) imply

$$\mathrm{d}q\,\mathrm{d}q = 0\,, \quad \mathrm{d}q\,\mathrm{d}p + \mathrm{d}p\,\mathrm{d}q = 0\,, \quad \mathrm{d}p\,\mathrm{d}p = 0\,. \tag{85}$$

As an involution we choose hermitean conjugation with $q^\dagger = q$, $p^\dagger = p$. A $\star$-operator satisfying the conditions (71) and (74) is determined by

$$\star\,1 = \mathrm{d}q\,\mathrm{d}p\,, \quad \star\,\mathrm{d}q = \mathrm{d}p\,, \quad \star\,\mathrm{d}p = -\mathrm{d}q\,, \quad \star\,(\mathrm{d}q\,\mathrm{d}p) = 1 \tag{86}$$

(so that $\epsilon_0 = \epsilon_2 = 1$, $\epsilon_1 = -1$). Now we consider a generalized harmonic map with values in the group of unitary elements U of $\mathcal{A}$ which satisfy $U^\dagger U = \mathbb{1} = U U^\dagger$. With

$$\begin{aligned} A &:= U^\dagger\,\mathrm{d}U \\ &= -\frac{1}{i\hbar}\,(U^\dagger p\,U - p)\,\mathrm{d}q + \frac{1}{i\hbar}\,(U^\dagger q\,U - q)\,\mathrm{d}p \end{aligned} \tag{87}$$

and thus

$$\star\,A = \frac{1}{i\hbar}\,(U^\dagger p\,U - p)\,\mathrm{d}p + \frac{1}{i\hbar}\,(U^\dagger q\,U - q)\,\mathrm{d}q \tag{88}$$

the field equation $\mathrm{d}\star A = 0$ becomes

$$[p, U^\dagger p\,U] + [q, U^\dagger q\,U] = 0\,. \tag{89}$$

In terms of $P := U^\dagger p\,U$, $Q := U^\dagger q\,U$ this takes the form

$$[p,P] + [q,Q] = -i\,\hbar\,(\hat{\partial}_q P - \hat{\partial}_p Q) = 0\,. \tag{90}$$

If $\mathrm{d}\alpha = 0$ for a 1-form α implies $\alpha = \mathrm{d}F$ with some $F \in \mathcal{A}$, then also (46) holds and all the required conditions are fulfilled. On the level of formal power series in q and p, every closed 1-form is indeed exact. Of course, one would like to substantiate these results on the level of functional analysis which, however, is beyond the scope of this work. $\diamondsuit$

4 Conclusions

We proposed a generalization of Riemannian geometry and harmonic maps within a wide framework of noncommutative geometry. It centers around a generalization of the Hodge operator to (noncommutative) differential calculi on associative algebras. The main motivation originated from previous work where we recovered lattice gauge theory from a

noncommutative geometry on $\mathbb{R}^n$ [6] and where we realized that, in particular, the nonlinear Toda lattice model can be expressed as an integrable harmonic map equation, but with respect to a simple deformation of the ordinary differential calculus [1-4] (cf the example in subsection 3.1).

In the present work we also addressed the case of 'noncommutative spaces'. Further work is certainly needed to understand the significance of 'noncommutative integrable models' and more examples have to be elaborated. It should also be possible to weaken the restrictions some more which we imposed above in order to generalize the classical construction of conservation laws for 2-dimensional σ-models to noncommutative spaces. The new material which we presented in subsection 3.3 has a fair chance to guide us to a class of interesting models in the same way as the intermediate step to noncommutative differential calculi on ordinary spaces led us [1] to physical models like the nonlinear Toda chain.

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